Project systems theory – Solutions

Resit exam 2016–2017, Wednesday 12 April 2017, 14:00 – 17:00

Problem 1

Consider the model

$$\dot{h}(t) = \frac{q_C(t) + q_H(t) - c\sqrt{h(t)}}{A}, \tag{1}$$

$$\dot{T}(t) = \frac{q_C(t)\big(T_C - T(t)\big) + q_H(t)\big(T_H - T(t)\big)}{Ah(t)}. \tag{2}$$

(a) For the desired equilibrium $h(t) = \bar{h}$, $T(t) = \bar{T}$, it holds that the time derivatives satsify $\dot{h} = \dot{T} = 0$. Substituting this in (1) leads to

$$0 = \bar{q}_c + \bar{q}_H - c\sqrt{\bar{h}}, \tag{3}$$

whereas (2) yields

$$0 = \bar{q}_C\big(T_C - \bar{T}\big) + \bar{q}_H\big(T_H - \bar{T}\big). \tag{4}$$

In the above equations, the constant inputs $q_C(t) = \bar{q}_C$ and $q_H(t) = \bar{q}_H$ are used. Given the desired equilibrium $\bar{h}$, $\bar{T}$, the corresponding inputs can be obtained from the linear set of equations

$$\begin{bmatrix} 1 & 1 \\ T_C - \bar{T} & T_H - \bar{T} \end{bmatrix} \begin{bmatrix} \bar{q}_C \\ \bar{q}_H \end{bmatrix} = \begin{bmatrix} c\sqrt{\bar{h}} \\ 0 \end{bmatrix}, \tag{5}$$

as derived from rewriting (3) and (4). Since $T_C < T_H$, it holds that $T_C - \bar{T} < T_H - \bar{T}$, such that the matrix on the left-hand side of (5) is nonsingular. Hence, there exists a *unique* constant input $q_C(t) = \bar{q}_C$, $q_H(t) = \bar{q}_H$.

In fact, the solutions are readily computed as

$$\bar{q}_C = \frac{c\sqrt{\bar{h}}(T_H - \bar{T})}{T_H - T_C}, \tag{6}$$

$$\bar{q}_H = \frac{c\sqrt{\bar{h}}(\bar{T} - T_C)}{T_H - T_C}, \tag{7}$$

from which it can also be observed that $\bar{q}_C \geq 0$ and $\bar{q}_H \geq 0$ for the equilibrium satisfying $\bar{h} > 0$ and $T_C \leq \bar{T} \leq T_H$.

(b) Define

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} h(t) \\ T(t) \end{bmatrix}, \qquad u(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} = \begin{bmatrix} q_C(t) \\ q_H(t) \end{bmatrix}, \tag{8}$$

and

$$f(x,u) = \begin{bmatrix} \frac{u_1 + u_2 - c\sqrt{x_1}}{A} \\ \frac{u_1(T_C - x_2) + u_2(T_H - x_2)}{Ax_1} \end{bmatrix}. \tag{9}$$

Then, the linearization of the dynamics (around the equilibrium $\bar{x}$ for constant input $\bar{u}$) is given by

$$\dot{\tilde{x}}(t) = \frac{\partial f}{\partial x}(\bar{x}, \bar{u})\tilde{x}(t) + \frac{\partial f}{\partial u}(\bar{x}, \bar{u})\tilde{u}(t), \tag{10}$$

where the perturbations are defined as

$$\tilde{x} = x - \bar{x}, \qquad \tilde{u} = u - \bar{u}. \tag{11}$$

Computation of the partial derivatives leads to

$$\frac{\partial f}{\partial x}(x, u) = \begin{bmatrix} -\frac{c}{2A\sqrt{x_1}} & 0 \\ -\frac{u_1(T_C - x_2) + u_2(T_H - x_2)}{Ax_1^2} & -\frac{u_1+u_2}{Ax_1} \end{bmatrix}, \qquad \frac{\partial f}{\partial u}(x, u) = \begin{bmatrix} \frac{1}{A} & \frac{1}{A} \\ \frac{T_c - x_2}{Ax_1} & \frac{T_H - x_2}{Ax_1} \end{bmatrix}, \tag{12}$$

which can be evaluated at the equilibrium to obtain

$$\tilde{A} = \frac{\partial f}{\partial x}(\bar{x}, \bar{u}) = \begin{bmatrix} -\frac{c}{2A\sqrt{\bar{h}}} & 0 \\ -\frac{\bar{q}_C(T_C - \bar{T}) + \bar{q}_H(T_H - \bar{T})}{A\bar{h}^2} & -\frac{\bar{q}_C + \bar{q}_H}{A\bar{h}} \end{bmatrix} = \begin{bmatrix} -\frac{c}{2A\sqrt{\bar{h}}} & 0 \\ 0 & -\frac{c}{A\sqrt{\bar{h}}} \end{bmatrix}, \tag{13}$$

$$\tilde{B} = \frac{\partial f}{\partial u}(\bar{x}, \bar{u}) = \begin{bmatrix} \frac{1}{A} & \frac{1}{A} \\ \frac{T_c - \bar{T}}{A\bar{h}} & \frac{T_H - \bar{T}}{A\bar{h}} \end{bmatrix}. \tag{14}$$

Then, the linearized dynamics is given as

$$\dot{\tilde{x}}(t) = \tilde{A}\tilde{x}(t) + \tilde{B}\tilde{u}(t), \tag{15}$$

with the matrices $\tilde{A}$ and $\tilde{B}$ as above. Note that these matrices are constant (as the equilibrium point is chosen).

(c) The stability is determined by the eigenvalues of A in (13), which equal its diagonal elements due to the diagonal structure of A. Since $c > 0$ and $\bar{h} > 0$, it is clear that the eigenvalues are real-valued and (strictly) negative, such that the system is (internally) stable.

Problem 2

Consider the family of polynomials

$$\mathcal{P}(\lambda) = \left\{ \lambda^3 + \alpha_2\lambda^2 + a\lambda + \alpha_0 \;\middle|\; a \leq \alpha_2 \leq 3a,\, 2a \leq \alpha_0 \leq 4a \right\}. \tag{16}$$

with a real.

Stability of a family of polynomials can be evaluated using Kharitonov's theorem, which states that all members of the family are stable if and only if four polynomials are stable. These four Kharitonov polynomials are given as

$$p_1(\lambda) = \lambda^3 + a\lambda^2 + a\lambda + 4a, \tag{17}$$
$$p_2(\lambda) = \lambda^3 + a\lambda^2 + a\lambda + 4a, \tag{18}$$
$$p_3(\lambda) = \lambda^3 + 3a\lambda^2 + a\lambda + 2a, \tag{19}$$
$$p_4(\lambda) = \lambda^3 + 3a\lambda^2 + a\lambda + 2a, \tag{20}$$

and it follows that $p_1 = p_2$ and $p_3 = p_4$. Hence, only stability of p_1 and p_3 needs to be checked.

Stability of $p_1 = p_2$ can be evaluated using the following Routh-Hurwitz table:

$$\begin{array}{r|cccc|l}
 & \lambda^3 & \lambda^2 & \lambda^1 & \lambda^0 & \\ \cline{2-5}
a\times & 1 & a & a & 4a & \\
1\times & a & & 4a & & \\ \cline{2-5}
 & & a^2 & a(a-4) & 4a^2 & \text{(step 1)} \\
(a-4)\times & & a & a-4 & 4a & \text{(after dividing by } a\text{, note } a \neq 0) \\
a\times & & a-4 & 0 & 0 & \\ \cline{3-5}
 & & & (a-4)^2 & 4a(a-4) & \text{(step 2)} \\
 & & & a-4 & 4a & \text{(after dividing by } a-4\text{, note } a-4 \neq 0)
\end{array}$$

This leads to the following conclusions. First, it is recalled that a necessary condition for stability of a polynomial is that all coefficients have the same sign. Thus, from the initial polynomial it immediately follows that $a > 0$. This condition also allows for division by a after step 1. Applying the same reasoning for the polynomial obtained at step 1 leads to the condition $a > 4$. Finally, checking stability of the polynomial that results from step 2 leads to

$$(a-4)\lambda + 4a = 0 \quad \Longrightarrow \quad \lambda = -\frac{4a}{a-4} < 0, \tag{21}$$

such that this polynomial is stable for $a > 4$ (i.e., the condition that was derived before). Thus, the polynomials p_1 and p_2 are stable if and only if $a > 4$.

Stability of $p_3 = p_4$ can be evaluated similarly. In this case, the Routh-Hurwitz table reads

$$\begin{array}{r|cccc|l}
 & \lambda^3 & \lambda^2 & \lambda^1 & \lambda^0 & \\ \cline{2-5}
3a\times & 1 & 3a & a & 2a & \\
1\times & 3a & & 2a & & \\ \cline{2-5}
 & & 9a^2 & a(3a-2) & 6a^2 & \text{(step 1)} \\
(3a-2)\times & & 9a & 3a-2 & 6a & \text{(after dividing by } a\text{, note } a \neq 0) \\
9a\times & & 3a-2 & 0 & 0 & \\ \cline{3-5}
 & & & (3a-2)^2 & 6a(3a-2) & \text{(step 2)} \\
 & & & 3a-2 & 6a & \text{(after dividing by } 3a-2\text{, note } 3a-2 \neq 0)
\end{array}$$

Following a similar reasoning as before (and by checking stability of the final polynomial), it follows that p_3 and p_4 are stable if and only if $a > \frac{2}{3}$.

Finally, application of Kharitonov's theorem ensures stability of the set of polynomials (16) whenever

$$a > 4. \tag{22}$$

Problem 3

Consider the system

$$\dot{x} = \begin{bmatrix} -1 & 2 & 0 \\ -2 & -1 & 1 \\ 0 & 0 & 2 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u. \tag{23}$$

(a) Controllability can be evaluated by computing the matrix

$$[\, B \;\; AB \;\; A^2B \,]. \tag{24}$$

Computing terms individually leads to

$$B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad AB = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \quad A^2B = A(AB) = \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix}, \tag{25}$$

such that

$$[\, B \;\; AB \;\; A^2B \,] = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 1 & 1 \\ 1 & 2 & 4 \end{bmatrix}. \tag{26}$$

Due to the triangular structure of the matrix, it is easy to see that

$$\operatorname{rank}[\, B \;\; AB \;\; A^2B \,] = 3, \tag{27}$$

such that (23) is controllable.

(b) Since (23) is controllable, there exists a nonsingular matrix T such that

$$T^{-1}AT = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \alpha_1 & \alpha_2 & \alpha_3 \end{bmatrix} = \bar{A}, \qquad T^{-1}B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \bar{B}. \tag{28}$$

for some real numbers α_1, α_2, and α_3. In fact, this form is known as the controllable canonical form and the parameters α_i equal the coefficients of the characteristic polynomial of A. As such, these can be computed by considering

$$\begin{aligned} \det\big(\lambda I - A\big) = (\lambda-2)\big((\lambda+1)(\lambda+1)+4\big) &= (\lambda-2)\big(\lambda^2+2\lambda+5\big), \\ &= \lambda^3 + \lambda - 10, \qquad (29) \\ &= \lambda^3 + a_1\lambda^2 + a_2\lambda + a_3, \qquad (30) \end{aligned}$$

with $a_1 = 0$, $a_2 = 1$, $a_3 = -10$. It then holds that

$$\alpha_1 = -a_3 = 10, \quad \alpha_2 = -a_2 = -1, \quad \alpha_3 = -a_1 = 0. \tag{31}$$

The corresponding transformation matrix T can be constructed by considering the vectors q_1, q_2, and q_3 as

$$q_3 = B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \tag{32}$$

$$q_2 = AB + a_1B = AB + 0B = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \tag{33}$$

$$q_1 = A^2B + a_1AB + a_2B = A^2B + 0AB + 1B = \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix}, \tag{34}$$

after which the matrix T can be constructed as

$$T = \begin{bmatrix} q_1 & q_2 & q_3 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 1 & 0 \\ 5 & 2 & 1 \end{bmatrix}. \tag{35}$$

Next, the condition $\bar{A} = T^{-1}AT$ can be checked without explicitly computing the inverse of T by verifying $T\bar{A} = AT$ instead. This leads to

$$AT = \begin{bmatrix} -1 & 2 & 0 \\ -2 & -1 & 1 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 1 & 1 & 0 \\ 5 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 2 & 0 \\ 0 & 1 & 1 \\ 10 & 4 & 2 \end{bmatrix}, \tag{36}$$

$$T\bar{A} = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 1 & 0 \\ 5 & 2 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 10 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 2 & 0 \\ 0 & 1 & 1 \\ 10 & 4 & 2 \end{bmatrix}, \tag{37}$$

which indeed verifies the desired result. Similarly,

$$T\bar{B} = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 1 & 0 \\ 5 & 2 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = B, \tag{38}$$

which verifies that $\bar{B} = T^{-1}B$ for the given transformation matrix T. Finally, note that the inverse of T is given as

$$T^{-1} = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ -\frac{3}{2} & -2 & 1 \end{bmatrix}. \tag{39}$$

(c) To place the eigenvalues of $A+BF$ at the locations -1, -1, and -2, consider the polynomial

$$(\lambda+1)^2(\lambda+2) = \left(\lambda^2+2\lambda+1\right)(\lambda+2) = \lambda^3+4\lambda^2+5\lambda+2, \tag{40}$$

such that the roots of this monic polynomial are the desired eigenvalues. Define

$$\bar{F} = \begin{bmatrix} \bar{F}_1 & \bar{F}_2 & \bar{F}_3 \end{bmatrix}, \tag{41}$$

and recall the definitions of $\bar{A} = T^{-1}AT$ and $\bar{B} = T^{-1}B$ in (28). Then, the closed-loop system matrix (in the "bar"-coordinates) is given as

$$\bar{A} + \bar{B}\bar{F} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 10+\bar{F}_1 & -1+\bar{F}_2 & \bar{F}_3 \end{bmatrix}, \tag{42}$$

which has the characteristic polynomial

$$\lambda^3 + (-\bar{F}_3)\lambda^2 + (1-\bar{F}_2)\lambda + (-10-\bar{F}_1) = 0. \tag{43}$$

Matching the coefficients of (40) and (43) leads to

$$\bar{F}_1 = -12, \quad \bar{F}_2 = -4, \quad \bar{F}_3 = -4. \tag{44}$$

After noting that

$$T(\bar{A}+\bar{B}\bar{F})T^{-1} = A + B\bar{F}T^{-1}, \tag{45}$$

it follows that the desired feedback (in the original coordinates) is given as $F = \bar{F}T^{-1}$. Solving the system of equations

$$FT = \begin{bmatrix} F_1 & F_2 & F_3 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 1 & 1 & 0 \\ 5 & 2 & 1 \end{bmatrix} = \bar{F} = \begin{bmatrix} -12 & -4 & -4 \end{bmatrix}, \tag{46}$$

leads to

$$F = \begin{bmatrix} 2 & 4 & -4 \end{bmatrix}. \tag{47}$$

Problem 4

Consider the system

$$\dot{x} = \begin{bmatrix} 6 & 1 & 0 \\ 0 & -3 & 0 \\ -8 & -1 & -2 \end{bmatrix} x + \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} u, \qquad y = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix} x. \tag{48}$$

(a) Stability is determined by the spectrum (i.e., the collection of eigenvalues) of A, which reads

$$\sigma(A) = \{6, -3, -2\}, \tag{49}$$

as can be concluded from the block-diagonal structure of A. Thus, the spectral radius reads

$$\Lambda(A) = \left\{ \max \mathrm{Re}(\lambda) \;\middle|\; \lambda \in \sigma(A) \right\} = 6 > 0, \tag{50}$$

and the system is not (internally) stable.

(b) The system is stabilizable if, for all $\lambda \in \sigma(A)$ such that $\mathrm{Re}(\lambda) \geq 0$, it holds that

$$\mathrm{rank} \begin{bmatrix} \lambda I - A & B \end{bmatrix} = 3. \tag{51}$$

Given the spectrum (49), this condition only needs to be evaluated for $\lambda = 6$. This leads to

$$\begin{bmatrix} 6I - A & B \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 & 2 \\ 0 & 9 & 0 & 0 \\ 8 & 1 & 8 & 1 \end{bmatrix}, \tag{52}$$

whose rank is indeed 3. Thus, the system is stabilizable.

(c) Observability can be checked by computing the observability matrix as

$$\begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ -2 & 0 & -2 \\ 4 & 0 & 4 \end{bmatrix}. \tag{53}$$

It holds that

$$\mathrm{rank} \begin{bmatrix} 1 & 0 & 1 \\ -2 & 0 & -2 \\ 4 & 0 & 4 \end{bmatrix} = 1 < 3, \tag{54}$$

such that the system is not observable.

(d) The system is detectable if

$$\mathrm{rank} \begin{bmatrix} \lambda I - A \\ C \end{bmatrix} = n = 3, \tag{55}$$

for all eigenvalues $\lambda \in \sigma(A)$ such that $\mathrm{Re}(\lambda) \geq 0$. A direct computation using $\lambda = 6$ as before leads to

$$\begin{bmatrix} \lambda I - A \\ C \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 9 & 0 \\ 8 & 1 & 8 \\ 1 & 0 & 1 \end{bmatrix}, \tag{56}$$

whose rank is readily seen to be 2. Thus, the system is not detectable.

(e) The unobservable subspace $\mathcal{N}$ is given as

$$\mathcal{N} = \ker \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix}. \tag{57}$$

It was concluded in problem (c) that the rank of the observability matrix equals 1, such that the dimension of the unobservable subspace is $3 - 1 = 2$. Using the result (53), it can be concluded that the unobservable subspace can be written as

$$\mathcal{N} = \operatorname{im} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \end{bmatrix}, \tag{58}$$

as the columns in the above matrix form a basis for the null space of the matrix in (53).

Problem 5

In order to prove that the characteristic equation of

$$M_n = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \ddots & & 0 \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ 0 & & & \ddots & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ -m_n & -m_{n-1} & -m_{n-2} & \cdots & -m_2 & -m_1 \end{bmatrix} \tag{59}$$

is given as

$$\Delta_M(\lambda) = \lambda^n + m_1\lambda^{n-1} + \ldots + m_{n-1}\lambda + m_n, \tag{60}$$

a proof by induction will be employed.

The base case is given for $k = 2$. Then, the characteristic equation is directly computed as

$$\Delta_{M_2}(\lambda) = \det(\lambda I_2 - M_2) = \begin{vmatrix} \lambda & -1 \\ m_2 & \lambda + m_1 \end{vmatrix} = \lambda(\lambda + m_1) + m_2 = \lambda^2 + m_1\lambda + m_2, \tag{61}$$

proving the desired result for $k = 2$.

Next, in the inductive step, assume that

$$\Delta_{M_{k-1}}(\lambda) = \lambda^{k-1} + m_1\lambda^{k-2} + \ldots + m_{k-2}\lambda + m_{k-1}. \tag{62}$$

Then, the characteristic equation for M_k is given as

$$\begin{aligned}\Delta_{M_k}(\lambda) &= \begin{vmatrix} \lambda & -1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & \lambda & -1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & \lambda & -1 & \ddots & & & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & & \vdots \\ 0 & 0 & & \ddots & \ddots & -1 & 0 & 0 \\ 0 & 0 & & & \ddots & \lambda & -1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & \lambda & -1 \\ m_k & m_{k-1} & m_{k-2} & m_{k-3} & \cdots & m_3 & m_2 & \lambda + m_1 \end{vmatrix} \\ &= \lambda \begin{vmatrix} \lambda & -1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & \lambda & -1 & \ddots & & & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & & \vdots \\ 0 & & \ddots & \ddots & -1 & 0 & 0 \\ 0 & & & \ddots & \lambda & -1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & \lambda & -1 \\ m_{k-1} & m_{k-2} & m_{k-3} & \cdots & m_3 & m_2 & \lambda + m_1 \end{vmatrix} + (-1)^{k-1} m_k \begin{vmatrix} -1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \lambda & -1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & \lambda & -1 & \ddots & & & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & & \vdots \\ 0 & & \ddots & \ddots & -1 & 0 & 0 \\ 0 & & & \ddots & \lambda & -1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & \lambda & -1 \end{vmatrix},\end{aligned}$$

where the second line is obtained by expanding the determinant using the first column. Because of the zeros in this column, this expansion has only two terms, corresponding to the minors with respect to the $(1,1)$ and $(m,1)$ elements in $\lambda I_k - M_k$. The first term has the same structure as the original matrix and it can be seen that this exactly represents the determinant of the matrix

$\lambda I_{k-1} - M_{k-1}$. Next, the determinant that appears in the second term is easily computed due to the lower triangular structure. Thus, it can be concluded that

$$\Delta_{M_k}(\lambda) = \lambda \det(\lambda I_{k-1} - M_{k-1}) + (-1)^{k-1} m_k \cdot (-1)^{k-1} \tag{63}$$
$$= \lambda \Delta_{M_{k-1}}(\lambda) + m_k \tag{64}$$
$$= \lambda^k + m_1 \lambda^{k-1} + \ldots + m_{k-1}\lambda + m_k, \tag{65}$$

where (62) is used to obtain (65).

Now, the base step (61) together with the inductive step (64) proves the desired result by induction.